Generating Functions in Combinatorics

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A Motivating Problem

Problem: A fish population starts out at 50 fish and grows 4-fold each year with 100 fish dying each year

Mathematical Formalism

- Population at time t is p_t
- Recurrence: $p_t = 4 \cdot p_{t-1} - 100$
- Base case: $p_0 = 50$

Natural question: What is p_t for any t?

A Fish population

Recurrence and Base Case: $p_t = 4 \cdot p_{t-1} - 100$, with $p_0 = 50$

Iterative Calculations

- $p_0 = 50$
- $p_1 = 100$
- $p_2 = 300$
- $p_3 = 1100$
- $p_4 = 4300$

We want a closed form!

Generating functions

A generating function takes a sequence of real numbers and makes it the coefficients of a formal power series.

Generating Function

Let $\{f_n\}_{n\geq 0}$ be a sequence of real numbers. Then the formal power series

$$F(x) = \sum_{n\geq 0} f_n x^n$$

is called the *ordinary generating function* of the sequence $\{f_n\}_{n\geq 0}$.

Formal Power Series

When using generating functions we will look at power series *formally*, meaning we *ignore convergence*.

Convergence

Consider the power series expansion

$$\frac{1}{1-x} = 1 + x + x^2 + \dots.$$

When $|x| < 1$, you can plug in x and the RHS $=$ LHS. For example, when $x = \frac{1}{2}$:

$$\frac{1}{1-1/2} = 2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots.$$

Formal Power Series Cont.

Example Cont.

$$\frac{1}{1-x} = 1 + x + x^2 + \dots.$$

When $|x| > 1$, plugging in x does not yield meaningful equalites. Consider $x = 2$:

$$\frac{1}{1-2} = -\frac{1}{2} \neq 1 + 2 + 4 + 8 + \dots = \infty.$$

Formal power series: Do not plug in values for x, because it is meaningless! We only care about the coefficients of the series.

Generating Functions for Solving Fish Population Problem

Define the generating function:

$$G(x) = \sum_{n \geq 0} p_n x^n.$$

First few terms: $G(x) = 50 + 100x + 300x^2 + \ldots$

Express Recurrence: $p_{t+1} = 4 \cdot p_t - 100$

$$\sum_{n \geq 0} p_{n+1} \cdot x^{n+1} = \sum_{n \geq 0} (4 \cdot p_n - 100) \cdot x^{n+1}$$

$$= \sum_{n \geq 0} 4 \cdot p_n \cdot x^{n+1} - \sum_{n \geq 0} 100 \cdot x^{n+1}$$

Solving Fish Population Problem Cont.

Generating Function equality:

$$\sum_{n\geq 0} p_{n+1} \cdot x^{n+1} = \sum_{n\geq 0} 4 \cdot p_n \cdot x^{n+1} - \sum_{n\geq 0} 100 \cdot x^{n+1}$$

- Left hand side: $G(x) - p_0$, since it's missing the first term of the sequence $\{p_n\}_{n\geq 0}$
- Right hand side term 1: $4x \cdot G(x)$
- Right hand side term 2: $-\frac{100x}{1-x}$, since $\frac{1}{1-x} = 1 + x + x^2 + \dots$

Recurrence in terms of $G(x)$:

$$G(x) - p_0 = 4x \cdot G(x) - \frac{100x}{1-x}$$

Solving Fish Population Problem Cont.

Want to solve following equation for closed form for p_t:

$$G(x) - p_0 = 4x \cdot G(x) - \frac{100x}{1-x}$$

After rearranging,

$$G(x) = \frac{p_0}{1-4x} - \frac{100x}{(1-x)(1-4x)}.$$

We have obtained an explicit formula for the $G(x)$, the generating function of the sequence $\{p_n\}$.

Finding formula for coefficients

Want closed form for coefficient of x^n in $G(x)$ because this is p_n.

$$G(x) = \frac{p_0}{1-4x} - \frac{100x}{(1-x)(1-4x)}.$$

First term's contribution is easy to calculate:

$$\frac{p_0}{1-4x} = 50\sum_{n\geq 0}(4x)^n = 50\sum_{n\geq 0} 4^n x^n$$

Finding formula for coefficients cont.

Expanding 2nd term yields confusion:

$$\frac{100x}{(1-x)(1-4x)} = 100x \cdot \sum_{n\geq 0} x^n \cdot \sum_{n\geq 0} 4^n x^n.$$

Another approach: partial fraction decomposition

We want to find constants A and B such that

$$\frac{100x}{(1-x)(1-4x)} = \frac{A}{1-x} + \frac{B}{1-4x}.$$

With $A = \frac{100}{3}$ and $B = \frac{-100}{3}$,

$$\frac{100x}{(1-x)(1-4x)} = \frac{100}{3} \cdot \frac{1}{1-4x} - \frac{100}{3} \cdot \frac{1}{1-x}.$$

Using Partial Fractions

$$\frac{100x}{(1-x)(1-4x)} = \frac{100}{3} \cdot \frac{1}{1-4x} - \frac{100}{3} \cdot \frac{1}{1-x}.$$

Expanding using power series yields:

$$\frac{100}{3} \cdot \frac{1}{1-4x} - \frac{100}{3} \cdot \frac{1}{1-x} = \frac{100}{3} \left(\sum_{n \geq 0} 4^n x^n - \sum_{n \geq 0} x^n \right).$$

Thus 2nd term's contribution to coefficient of x^n is:

$$\frac{100}{3}(4^n - 1).$$

An explicit formula for p_n

Recall

$$G(x) = \frac{p_0}{1-4x} - \frac{100x}{(1-x)(1-4x)}.$$

First term's contribution:

$$50 \cdot 4^n.$$

Second term's contribution:

$$\frac{100}{3}(4^n - 1).$$

Combining contributions, closed-form formula for p_n is:

$$p_n = 50 \cdot 4^n - 100 \cdot \frac{4^n - 1}{3}.$$

Exponential Generating Functions

Exponential generating functions are every similar to ordinary generating functions.

Exponential Generating Function

Let $\{f_n\}_{n\geq 0}$ be a sequence of real numbers. Then the formal power series

$$F(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!},$$

is called the *exponential generating function* of the sequence $\{f_n\}_{n\geq 0}$.

Intuition: Dividing by $n!$ allows for f_n to grow faster.

Motivating Example

Recurrence Relation: Solve for a_n if $a_0 = 1$, and a_n satisfies the following recurrence

$$a_{n+1} = (n+1)(a_n - n + 1).$$

First few terms

- $a_0 = 1$
- $a_1 = 2$
- $a_2 = 4$
- $a_3 = 9$
- $a_4 = 28$
- $a_5 = 125$

This series grows too fast for an ordinary generating function. Therefore an exponential generating function is used.

Solving recurrence with exponential generating functions

Defining generating function:

$$A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!},$$

is the exponential generating function of the sequence $\{a_n\}_{n\geq 0}$.

Expressing recurrence $a_{n+1} = (n+1)(a_n - n + 1)$:

$$\sum_{n=0}^{\infty} a_{n+1} \frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n!} - \sum_{n=0}^{\infty} (n-1) \frac{x^{n+1}}{n!}.$$

Solving recurrence cont.

$$\sum_{n=0}^{\infty} a_{n+1} \frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n!} - \sum_{n=0}^{\infty} (n-1) \frac{x^{n+1}}{n!}.$$

- LHS $= A(x) - 1$
- RHS first term: $xA(x)$
- RHS second term: $-x^2e^x + xe^x = (x - x^2)e^x$

Plugging in above:

$$A(x) - 1 = xA(x) - x^2e^x + xe^x.$$

Rearranging yields,

$$A(x) = \frac{1}{1-x} + xe^x.$$

Thus coefficient a_n for $\frac{x^n}{n!}$ is $a_n = n! + n$.

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References

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The End